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Quantum derivation of the excess noise factor in lasers with non-orthogonal eigenmodes

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Abstract. By generalizing recently obtained results we calculate the excess noise factor (Petermann factor) for a laser system with non-orthogonal eigenmodes. The quantum consistency of the calculation is shown through the explicit conservation of input-output commutation rules. As a result of the calculation, the excess noise in the lasing mode is shown to depend on the laser gain below threshold, and on the noise analysis frequency below and above threshold.

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1 Introduction

Several theoretical and experimental studies have been recently devoted to the excess quantum noise which appears in lasers with non-orthogonal eigenmodes. One striking feature of such systems is that the laser linewidth can be much larger than the usual Schawlow-Townes linewidth, by a factor which is known as the Petermann excess noise factor K [1–5]. In recent experiments, excess noise factors as large as several hundreds have been reported in lasers with non-orthogonal transverse [6–9] or polarization [10] modes.

From a theoretical point of view, the essential feature of these laser systems is their non-Hermitian character, leading to non-orthogonal eigenmodes [2]. In particular, losses due to the aperturing within the cavity play an essential role in the definition of transverse laser modes [6–9]. This feature has very important consequences. First, the relation between input and output laser modes in a cavity round trip is non-unitary, and it is also non-invariant under propagation reversal. Second, as it was analyzed in detail by Siegman [2], one can define laser eigenmodes as modes which reproduce after one round trip in the cavity, up to a complex multiplicative constant. Then the set of laser eigenmodes $\{u_n\}$ is non-orthogonal, but it is bi-orthogonal to another "adjoint" set of modes $\{v_n\}$, where the set $\{v_n^*\}$ is obtained by reverting the direction of propagation in the cavity. One has therefore:

$$(u_n, u_n) = 1, \qquad (u_n, v_m) = \delta_{nm}\,, \tag{1}$$

where (,) denotes the Hermitian scalar product. It follows from equation (1) that the modes of the set $\{v_n\}$ cannot be normalized. For a given resonator, the set of modes $\{u_n\}$ and $\{v_n\}$ can be calculated numerically; it is then straightforward to calculate K, which is simply given for mode n by [5,6]

$$K_n = (v_n, v_n)\,. \tag{2}$$

Though this picture works quite efficiently, and is in very good agreement with the experiments, it has some built-in conceptual difficulties. The main one is how to turn this semi-classical model into a fully quantum description: the complex amplitudes of a set of classical non-orthogonal modes cannot be turned into a set of non-commuting operators [3], because of problems related to unitarity (such a procedure would violate the conservation of probability). In a recent paper [11], we showed on a simple "toy model" that this difficulty can in principle be solved by introducing appropriate "vacuum modes" [4], that allow one to recover the unitarity of the input-output scattering matrix [12,13]. The purpose of the present paper is to generalize this approach to a laser with an arbitrary mode configuration. As in reference [11], the laser cavity is described using a scattering matrix, which appears to be non-unitary when restricted to the subset of laser modes, while it is unitary (as it should be expected) when the set is extended to include modes in loss and gain channels.

The paper is organized as follow: in Section 2, we consider the case of a single mode laser, and we present some useful results about the respective contributions of the "gain" and "loss" mechanisms to the value of the laser linewidth. In Section 3, we use these results, and some algebra derived from the semi-classical approach by Siegman in reference [14], in order to calculate the laser excess noise in a fairly general and fully quantum approach. We do recover the usual expression of the Petermann

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factor above threshold, but the excess noise is found to depend on the laser gain below threshold, and on the noise analysis frequency both below and above threshold.

2 Calculating the laser phase noise

We consider the round-trip equation describing a single mode laser close to threshold. During one round trip, the mode will experience losses γ and gain g, which compensates each other so that the value of the net gain γg is equal to one. The generic transformation of the laser mode operator a along a cavity round trip will have thus the following form:

$$\hat{a}_{\rm out} = \gamma g \hat{a}_{\rm in} + \sum_i \alpha_i \hat{b}_{{\rm vac},i} + \sum_j \beta_j \, \hat{b}^\dagger_{{\rm sp},j}, \tag{3}$$

where the $\hat{b}_{{\rm vac},i}$ are "vacuum noise" operators associated to the losses during the cavity round trip, while the $\hat{b}_{{\rm sp},j}$ are noise operators corresponding to the spontaneous emission of the amplifier. We assume that the i and j modes are distinct, so that $\hat{b}$ and $\hat{b}^\dagger$ never appear together for the same mode (this would correspond to a phase-dependent mechanism, such as phase conjugation, which does not occur usually in a laser amplifier). The normalization of the coefficients α_i, β_j is chosen so that all modes obey standard commutation relations $[\hat{b}_l, \hat{b}^\dagger_m] = \delta_{lm}$. Since the input-output transformation given by equation (3) must be unitary, *i.e.* preserve the commutation relations, one must have $[\hat{a}_{\rm out}, \hat{a}^\dagger_{\rm out}] = [\hat{a}_{\rm in}, \hat{a}^\dagger_{\rm in}] = 1$, so that

$$1 = |\gamma g|^2 + \sum_i |\alpha_i|^2 - \sum_j |\beta_j|^2 \,. \tag{4}$$

Assuming, as said above, that for a lasing mode one has $\gamma g = 1$, one obtains thus

$$\sum_i |\alpha_i|^2 = \sum_j |\beta_j|^2 \,. \tag{5}$$

This equation establishes a relation between the noises due to the cavity loss ("vacuum" noise in the i modes) and the cavity gain (spontaneous emission noise in the j modes), which will turn out to be useful in the following.

In order to deal with the laser phase noise, it is convenient to define for each mode the phase quadrature operator $Y = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$; the corresponding mode will be indicated by the same lower index as in equation (3). To obtain the expression of the laser linewidth, we express the variation of the source term $\delta\Gamma_Y$ for the laser phase noise during a round-trip time $\delta t = \tau_{\rm rt} = 2L/c$ (c is the speed of light, and $2L$ the total cavity length):

$$\begin{aligned} \tau_{\rm rt} \frac{\delta\Gamma_Y}{\delta t} &= Y_{\rm out} - Y_{\rm in} \\ &= \sum_i \alpha_i Y_{{\rm vac},i} + \sum_j \beta_j Y_{{\rm sp},j} \,. \end{aligned} \tag{6}$$

It should be mentioned that equation (6) is valid above threshold, owing to the well-verified approximation that phase noise experiences no restoring force from the gain saturation mechanism. On the other hand, a correct handling of the amplitude noise, described by the amplitude quadrature operators $X = \frac{1}{2}(a + a^\dagger)$, would have to take into account the gain saturation mechanism, which is not the purpose of the present paper.

Using standard calculations, the laser linewidth above threshold is given as a function of the (symmetrically ordered) variance of $\delta\Gamma_Y$ as

$$\Delta\omega = \left(\frac{c}{2L}\right) \frac{\langle \delta\Gamma_Y^2 \rangle}{4|\langle a_{\rm out}\rangle|^2} \,, \tag{7}$$

where $|\langle a_{\rm out}\rangle| = |\langle a_{\rm in}\rangle|$ is the mean value of intracavity field of the lasing mode. We will assume in the following that all fields can be considered constant within the laser cavity (mean-field approximation); deviations from this approximation lead to the so-called "longitudinal Petermann factor" [11,15]. Using $\langle Y^2_{{\rm vac},i}\rangle = \langle Y^2_{{\rm sp},j}\rangle = 1/4$ for each mode in the vacuum state, one obtains

$$\langle \delta\Gamma_Y^2 \rangle = \frac{1}{4} \left(\sum_i |\alpha_i|^2 + \sum_j |\beta_j|^2 \right) . \tag{8}$$

From this expression and equation (5), it is clear that $\langle \delta\Gamma_Y^2 \rangle$ can be split into two equal parts:

$$\langle \delta\Gamma_{\rm vac}^2 \rangle = \frac{1}{4} \left(\sum_i |\alpha_i|^2 \right), \quad \langle \delta\Gamma_{\rm sp}^2 \rangle = \frac{1}{4} \left(\sum_j |\beta_j|^2 \right) . \tag{9}$$

Therefore, the laser linewidth can be written as

$$\Delta\omega = \left(\frac{c}{2L}\right) \frac{\langle \delta\Gamma_{\rm sp}^2 \rangle}{2|\langle a_{\rm out}\rangle|^2} = \left(\frac{c}{2L}\right) \frac{\langle \delta\Gamma_{\rm vac}^2 \rangle}{2|\langle a_{\rm out}\rangle|^2}. \tag{10}$$

As often occurs in quantum mechanics [3,16], one has the choice to attribute the linewidth either completely to the gain mechanism (this would correspond to a "normal" ordering of the operators), or to the loss mechanism (this would correspond to an "antinormal" ordering of the operators), or to split half and half ("symmetrical" ordering, which is preferred here). In this last point of view, the noise in a laser *above threshold* has two equally contributing origins, namely the vacuum modes and the amplifier.

As an example, let us consider the simple case where a single mode experiences a gain g and round-trip reflexion r, such that $gr = 1$. One has then

$$\hat{a}_{\rm out} = gr \, \hat{a}_{\rm in} + g\sqrt{1 - |r|^2} \;\, \hat{b}_{\rm vac} + \sqrt{|g|^2 - 1} \;\, \hat{b}^\dagger_{\rm sp} \,. \tag{11}$$

It is straightforward to see that the two phase noise terms have an equal contribution $\langle \delta\Gamma_{\rm sp}^2 \rangle = \langle \delta\Gamma_{\rm vac}^2 \rangle = \frac{1}{4}(|g|^2 - 1)$ (since $|gr| = 1$). Since this value leads directly to the standard Schawlow-Townes value for the laser linewidth (in the mean field approximation [17]), we will consider in the next section that $\langle \delta\Gamma^2_{\rm ST}\rangle = \frac{1}{4}(|g|^2 - 1)$ as the reference value, with respect to which the "excess linewidth" will be evaluated. The so-called "Petermann factor" will thus appear as the ratio between the actual value of $\langle \delta\Gamma^2_{\rm vac}\rangle$ for the lasing mode and the reference value $\langle \delta\Gamma^2_{\rm ST}\rangle$.

3 Derivation of the Petermann excess noise factor

3.1 Framework of the calculation

Our purpose now is to generalize the result of the previous section to a single mode laser with a multimode cavity structure, in order to calculate explicitly the Petermann excess noise factor. This calculation will apply for instance to the multimode transverse structure of a (stable or unstable) laser cavity with strong aperturing losses. We will consider the situation of a homogeneously broadened gain medium, where all cavity modes essentially see the same gain medium [18]. The gain value is therefore the same for all modes, while on the other hand separate and independent spontaneous emission noise operators are attributed to each mode. As shown in reference [11], the essential new feature which will create excess noise is a redistribution of the noise between the different laser modes, which is carried out by loss modes which are common to several laser modes. As a matter a fact, by generalizing the mathematical formalism introduced in reference [11], the present results confirm that "loss-induced coupling", *i.e.* the fact that laser modes are coupled by sharing noise due to common loss modes, is central to the issue of excess phase noise (see also the Appendix).

As a last point about the scope of the present calculation, we point out that our whole approach is based upon the linearized input-output formalism which is standard in quantum optics [11–13]. This formalism will be valid either below threshold, in the linear gain regime, or above threshold, where the quantum fluctuations can be linearized around the semi-classical values. In the latter case, our model gives a correct description of the (relatively small) quantum fluctuations around the semi-classical values. On the other hand, it is well known that such a linearized approach breaks down right on singularity points. In the steady state of macroscopic lasers, this is generally not a limitation, because quantum fluctuations are very small in relative value, and they may increase a lot before the linearization actually breaks down. Taking into account the above restrictions, the calculations presented below will be used to get predictions both below and above threshold.

3.2 Classical non-orthogonal modes

Let us consider first the multimode cavity structure without the gain mechanism ("cold cavity" situation). For quantum consistency, the round-trip equation should include not only the "laser" modes, which will see the gain, but also the "vacuum" modes which correspond to the various loss channels. We introduce therefore a set of normalized and orthogonal (classical) mode functions, which correspond to all input modes into the system. Any mode can be decomposed using this set as a basis, and will be written as a column vector $\{e_{\rm in}\}$ (input modes) or $\{e_{\rm out}\}$ (output modes). For instance, the n-th basis vector is represented by a column with 1 on the n-th line and 0 everywhere else. The general input-output transformation for the cold cavity can then be written:

$$\{e_{\rm out}\} = S\{e_{\rm in}\}. \tag{12}$$

Since the set of mode functions will be used later on as a quantization basis, the scattering matrix S is by definition unitary, in order to insure that all operator commutation relations will be preserved in the input-output evolution. Since the modes can be split into two sets of "laser" and "loss" modes, it is convenient to introduce (Hermitian) projection operators P and Q, such as

$$P^2 = P, \quad Q^2 = Q, \quad P + Q = 1\,, \tag{13}$$

where P projects on the "laser" modes subset, and Q on the "loss" modes subset. One obtains therefore

$$P\{e_{\rm out}\} = PS\,(P+Q)\{e_{\rm in}\} = TP\{e_{\rm in}\} + PSQ\{e_{\rm in}\}, \tag{14}$$

where $PSQ\{e_{\rm in}\}$ corresponds to the contribution of the loss modes, while the "truncated" scattering matrix $T = PSP$ describes the input-output transformation for the laser modes only. In general, T is no more unitary, but one can find its eigenvalues and eigenvectors under the form [14]

$$TU = UG\,, \tag{15}$$

where U corresponds to a matrix with columns formed by the normalized eigenvectors $\{u_n\}$ of T, and G to a diagonal matrix formed by the corresponding eigenvalues γ_n. In general, the eigenvectors of T are non-orthogonal (see again ref. [11] for a simple illustrative example), and therefore U is not unitary. More precisely, we note that $U^\dagger U = I + J$, where I is the unit matrix, and J is a purely non-diagonal Hermitian operator.

Defining $V = (U^{-1})^\dagger$, and multiplying both sides of equation (15) by $V^\dagger$, one obtains $V^\dagger T = GV^\dagger$, and therefore

$$T^\dagger V = VG^\dagger\,. \tag{16}$$

This equation shows that V corresponds to a matrix with columns formed by the eigenvectors $\{v_n\}$ of $T^\dagger$, while $G^\dagger$ is a diagonal matrix formed by the corresponding eigenvalues γ_n^* [19]. One obtains immediately from the definition of V that, given the normalized eigenvectors $\{u_n\}$, the vectors $\{v_n\}$ cannot be normalized, but that one has ("bi-orthogonality" relations):

$$V^\dagger U = U^\dagger V = I\,. \tag{17}$$

Then, by applying the operator $PV^\dagger = PU^{-1}$ to equation (14), and using $V^\dagger T = GV^\dagger$, one obtains

$$PV^\dagger P\{e_{\rm out}\} = G\left(PV^\dagger P\{e_{\rm in}\}\right) + PV^\dagger PSQ\{e_{\rm in}\}\,. \quad (18)$$

This operation (projection on the bi-orthogonal) corresponds to moving into a basis where the in-out transformation for the laser modes is diagonal. However, this basis, which is made from the vectors $\{v_n\}$, is both non-normalized and non-orthogonal, and is not appropriate for quantization; we will come back to that in detail below.

3.3 Operatorial equations for a cavity round trip

Up to now, we considered only classical mode functions, so that all modes vectors were c-numbers. We will see now how to introduce the mode operators. In the initial orthonormal basis, a quantized field operator can be written simply as $\hat{A} = \{\hat{a}\}$, where each operator $\hat{a}_n$ on the n-th line corresponds to the destruction of one photon in one mode of the basis set. One has thus $\{\hat{a}_{\rm out}\} = S\{\hat{a}_{\rm in}\}$, and it can be checked easily that this transformation preserves all commutation relations, due to the unitarity of S. Then the mode operator associated to the normalized eigenvector u_n will be the linear combination $(PU^\dagger P\{\hat{a}\})_n$, which insures that the operator $(\{\hat{a}\}^\dagger PUP)_n$ (here n is a column index) creates a photon in mode u_n. Though each of these combinations taken separately is acceptable as a mode operator, the $(PU^\dagger P\{\hat{a}\})_n$ taken all together are not acceptable as a *set* of mode operators, due to the non-orthogonality of the u_n vectors. As a matter of fact, it can be checked easily that the commutator between $(PU^\dagger P\{\hat{a}\})_n$ and $(\{\hat{a}\}^\dagger PUP)_m$ is simply the scalar product $(u_n,\ u_m)$, which is one for $n = m$, but is generally non-zero if $n \neq m$. The alternative quantity $(PV^\dagger P\{\hat{a}\})_n$ is not acceptable either as a mode operator, again because it does not fulfill basic commutation rules. We note that this behaviour, due to the non-unitarity of T, is very different from the one encountered for a unitary operator, such as S itself. Denoting the matrices of eigenvalues and eigenvectors of S with a subscript S, the same procedure would yield

$$SU_S = U_S G_S, \quad V_S = (U_S^{-1})^\dagger = U_S\,,$$
$$U_S^\dagger\{\hat{a}_{\rm out}\} = U_S^\dagger S\{\hat{a}_{\rm in}\} = G_S U_S^\dagger\{\hat{a}_{\rm in}\}\,. \quad (19)$$

It is thus clear that the set of operators $U_S^\dagger\{a\}$ correspond to the normalized and orthogonal eigenmodes of S, while neither $PU^\dagger P\{\hat{a}\}$ nor $PV^\dagger P\{\hat{a}\}$ can play this role for T. We will show below that an acceptable operator equation can nevertheless be obtained, by considering *separately* the particular operator which corresponds to the lasing mode.

The next step is to introduce the gain mechanism. From the hypothesis discussed above, the gain matrix is $gP+Q$, and independent spontaneous emission noise adds up to each of the initial laser modes [20]. Equation (14) can thus be written in the operatorial form:

$$P\{\hat{a}_{\rm out}\} = g\left(TP\{\hat{a}_{\rm in}\} + PSQ\{\hat{a}_{\rm in}\}\right) + \\ +\sqrt{|g|^2-1}\;P\{\hat{b}_{\rm sp}^\dagger\}\,, \quad (20)$$

where $P\{\hat{b}_{\rm sp}^\dagger\}$ is a column vector of spontaneous emission noise operators, each one corresponding to one mode of the "laser" subset. Equation (18) can then be written as

$$P\ \ V^\dagger P\{\hat{a}_{\rm out}\} = gG\left(PV^\dagger P\{\hat{a}_{\rm in}\}\right) \\ +gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}_{\rm sp}^\dagger\}. \quad (21)$$

This equation provides the basis for the excess noise calculation shown below.

3.4 Calculation of the excess noise

Generally speaking, the quantity $\{a_{\rm out}\} - \{a_{\rm in}\} = (S - I)\{a_{\rm in}\} = \delta(\{a_{\rm in}\})$ represents the variation of the mode operators during a round trip. For the modes of the laser subset, considered first below threshold, equation (21) can be used to obtain the fluctuations in the steady state, by writing that $\delta(P\{a_{\rm in}\}) = P\{a_{\rm out}\} - P\{a_{\rm in}\} = 0$ during a round trip [11]. This equality is actually valid for fluctuations at zero frequency, and a more detailed analysis of the fluctuation spectrum will be considered in Section 3.5 below. One obtains thus

$$(I-gG)PV^\dagger P\ \{\hat{a}_{\rm out}\} = \\ gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}_{\rm sp}^\dagger\}\,. \quad (22)$$

Taking the m-th line of equation (22), one obtains for any m:

$$(PV^\dagger P\ \{\hat{a}_{\rm out}\})_m = \frac{1}{1-g\gamma_m} \\ \times\left(gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}_{\rm sp}^\dagger\}\right)_m. \quad (23)$$

Let us consider now the mode with the smallest losses, which will be called the "lasing mode" and denoted by the index n. It appears that as g approaches its threshold value $1/\gamma_n$, the behaviour of the lasing and non-lasing modes will be quite different. For non-lasing modes, the effective gain $1/(1-g\gamma_m)$ will remain finite, so that the zero-frequency noise in the mode can be obtained from equation (23). For the lasing mode, the effective gain $1/(1-g\gamma_n)$ will diverge, until it is limited by saturation effects. Then the source term for the noise in the lasing mode will be obtained as the variation of the corresponding operator during a round trip, as shown in Section 2. Using as previously the notation $Y = \frac{1}{2i}(\hat{a}-\hat{a}^\dagger)$, the source term for the phase noise term in the lasing mode can still be written:

$$\tau_{\rm rt}\frac{\delta\Gamma_Y}{\delta t} = (PU^\dagger P\{Y_{\rm out}\})_n - (PU^\dagger P\{Y_{\rm in}\})_n\ . \quad (24)$$

This quantity can be obtained by using the identity $V^\dagger = U^\dagger + (I - U^\dagger U)V^\dagger$, which implies:

$$(PU^\dagger P\{\hat{a}_{\rm out}\})_n = (PV^\dagger P\{\hat{a}_{\rm out}\})_n \\ +(P(U^\dagger U - I)PV^\dagger P\ \{\hat{a}_{\rm out}\})_n. \quad (25)$$

Since $(U^\dagger U - I)$ is purely non-diagonal, the expression $(P(U^\dagger U - I)PV^\dagger P\ \{\hat{a}_{\rm out}\})_n$ involves the values of $(PV^\dagger P\ \{\hat{a}_{\rm out}\})_m$ with $m \neq n$. These terms can be deduced from equation (23), and they are not diverging when approaching the threshold. On the other hand, it is more convenient for the lasing mode to come back to equation (21), which has no diverging term. Inserting equation (25) in the n-th line of equation (21), one obtains the round-trip evolution for the lasing mode

$$\begin{aligned}(PU^\dagger P\ \{\hat{a}_{\rm out}\})_n &= g\gamma_n\ (PU^\dagger P\ \{\hat{a}_{\rm in}\})_n \\ &+ \Big(gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\Big)_n \\ &+(1-g\gamma_n)\ \big(P(U^\dagger U - I)\ (I-gG)^{-1} \\ &\times \Big[gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\Big]\Big)_n . \quad (26)\end{aligned}$$

Assuming now that the gain g is such that $g\gamma_n = 1$, one obtains from equations (24) and (26):

$$\begin{aligned}\tau_{\rm rt}\frac{\delta\Gamma_Y}{\delta t} &= (1/\gamma_n)\ (PV^\dagger PSQ\{Y_{\rm in}\})_n \\ &+\sqrt{1/|\gamma_n|^2-1}\ (PV^\dagger P\{Y_{\rm sp}\})_n\,. \quad (27)\end{aligned}$$

One recovers therefore *for the lasing mode* the generic relation given by equation (6).

3.5 Value of the excess noise factor at threshold

As shown in the previous section, we can now obtain the laser linewidth by considering either the "vacuum" or the "spontaneous emission" noise term in equation (27). For the sake of completeness, let us calculate both by introducing the column matrices:

$$\begin{aligned}P\{\delta\Gamma_{\rm vac}\} &= (1/\gamma_n)\ (PV^\dagger PSQ\{Y_{\rm in}\})\,, \\ P\{\delta\Gamma_{\rm sp}\} &= \sqrt{1/|\gamma_n|^2-1}\ (PV^\dagger P\{Y_{\rm sp}\})\,. \quad (28)\end{aligned}$$

The corresponding covariance matrices are then

$$\begin{aligned}&P\langle\{\delta\Gamma_{\rm vac}\}\{\delta\Gamma_{\rm vac}\}^\dagger\rangle P = \\ &\quad (PV^\dagger PSQ)V_{\rm in}(Q^\dagger S^\dagger P^\dagger VP^\dagger)\ /|\gamma_n|^2, \\ &P\langle\{\delta\Gamma_{\rm sp}\}\{\delta\Gamma_{\rm sp}\}^\dagger\rangle P = \\ &\quad (PV^\dagger P)\ V_{\rm in}\ (P^\dagger VP^\dagger)(1/|\gamma_n|^2-1)\,, \quad (29)\end{aligned}$$

where $V_{\rm in}$ is a diagonal matrix with all elements equal to $\langle{\rm vac}|Y^2_{\rm in}|{\rm vac}\rangle = \frac{1}{4}$. Using $SS^\dagger = 1$ and the properties of the projection operators, one obtains for the first covariance matrix:

$$\begin{aligned}&P\langle\{\delta\Gamma_{\rm vac}\}\{\delta\Gamma_{vac}\}^\dagger\rangle P = \\ &\quad \big(PV^\dagger PS\ (1-P)\ S^\dagger PVP\big)/(4|\gamma_n|^2) = \\ &\quad \big(PV^\dagger VP - PV^\dagger TT^\dagger VP\big)/(4|\gamma_n|^2) = \\ &\quad \big(PVV^\dagger P - PGV^\dagger VG^\dagger P\big)/(4|\gamma_n|^2)\,, \quad (30)\end{aligned}$$

while the second one is simply

$$\begin{aligned}&P\ \ \langle\{\delta\Gamma_{\rm sp}\}\{\delta\Gamma_{\rm sp}\}^\dagger\rangle P = \\ &\quad \big(PV^\dagger VP\big)\,(1-|\gamma_n|^2)/(4|\gamma_n|^2)\,. \quad (31)\end{aligned}$$

Taking the variance of the n-th mode (n-th component on the diagonal of the covariance matrix), and dividing by the reference value $\langle\delta\Gamma^2_{\rm ST}\rangle = \frac{1}{4}(1/|\gamma_n|^2-1)$, one obtains two expressions for the excess noise factor for mode n:

$$\begin{aligned}K_{{\rm vac},n} &= \frac{(V^\dagger V - GV^\dagger VG^\dagger)_{n,n}}{1-|\gamma_n|^2}, \\ K_{{\rm sp},n} &= (V^\dagger V)_{n,n}. \quad (32)\end{aligned}$$

Since G is a diagonal matrix, it is easy to show that $1-|\gamma_n|^2$ actually cancels out between the numerator and denominator of $K_{{\rm vac},n}$, so that both values end up to be $K_n = (V^\dagger V)_{n,n}$. This value is just the squared modulus of the biorthogonal vector v_n for mode n, which is the standard result [5,6].

Finally, we point out that as long as the laser is below threshold (*i.e.* $g\gamma_n < 1$), the step of the calculation going from equation (26) to equation (27) cannot be carried out; correspondingly, equation (5) is not valid, and the full equation (26) should be used to obtain the excess noise. An example of such calculation was presented in reference [11], in the simple case of a two-mode problem. As a general rule, the excess spontaneous emission noise will decrease with g, down from the maximum value K obtained at threshold. As a consequence, it is not correct to assume that the spontaneous emission noise in the lasing mode is enhanced by a factor K, irrespective of the gain value. Though it can be evaluated from the properties of the "cold cavity" alone, the excess noise factor gets its precise meaning only at or above threshold. It may be worth pointing out also that below threshold there is no more semi-classical field acting as a local oscillator to pin out the lasing mode. Therefore, though equation (26) remains valid in principle, an operational definition of how to measure the "noise in the lasing mode" would be needed.

3.6 Frequency spectrum of the excess noise

The importance of the last term in equation (26) can also be illustrated by looking at the noise spectrum of the fluctuations. For fluctuations at non-zero frequency, the round-trip equation $\delta(P\{a_{\rm in}\}) = 0$ which was used before should be replaced by $\tau_{\rm rt}\frac{\rm d}{{\rm d}t}P\{a_{\rm in}\} = P\{a_{\rm out}\} - P\{a_{\rm in}\}$. It is actually more convenient to work in frequency space, where the time derivative becomes $i\Omega P\{a_{\rm in}\}$, $\Omega = \omega\tau_{\rm rt}$ being a dimensionless frequency. The previous equations thus remain valid, by simply changing $(1-g\gamma_m)$ into $(1-g\gamma_m+i\Omega)$. Equation (26) thus becomes

$$\begin{aligned}(PU^\dagger P\ \{\hat{a}_{\rm out}\})_n &= g\gamma_n\ (PU^\dagger P\ \{\hat{a}_{\rm in}\})_n \\ &+ \Big(gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\Big)_n \\ &+(1-g\gamma_n+i\Omega)\ \big(P(U^\dagger U - I)\ (I-gG+i\Omega I)^{-1} \\ &\times \Big[gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\Big]\Big)_n \quad (33)\end{aligned}$$

and by taking $g\gamma_n = 1$:

$$(PU^\dagger P\{\hat{a}_{\rm out}\})_n = (PU^\dagger P\{\hat{a}_{\rm in}\})_n + \left(gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\right)_n + i\Omega\left(P(U^\dagger U - I)(I - gG + i\Omega I)^{-1} \times \left[gPV^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PV^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\right]\right)_n. \quad (34)$$

This equation shows that the excess quantum noise is frequency dependent. Since the case $\Omega = 0$ was treated above, it is particularly interesting to consider now the high-frequency case $\Omega \gg (1 - \gamma_m/\gamma_n)$ for all $m \neq n$, so that $i\Omega(I - gG + i\Omega I)^{-1} = I$. Then the term which was previously responsible for the K factor cancels out, and using $UV^\dagger = I$, one obtains

$$(PU^\dagger P\{\hat{a}_{\rm out}\})_n = (PU^\dagger P\{\hat{a}_{\rm in}\})_n + \left(PU^\dagger PSQ\{\hat{a}_{\rm in}\} + \sqrt{|g|^2-1}PU^\dagger P\{\hat{b}^\dagger_{\rm sp}\}\right)_n, \quad (35)$$

which is just the usual equation for the lasing mode in the presence of gain and loss: the excess noise has thus completely disappeared. We note that high values of K correspond to the existence of at least one eigenvalue γ_m close to γ_n (see again [11]). In that case, the normalized cut-off frequency $(1 - g\gamma_m) = (1 - \gamma_m/\gamma_n)$, is also very small: it appears therefore that the highest the excess noise, the smallest its bandwidth. This bandwidth corresponds simply to the effective damping of the mode nearest to threshold, which gives the highest contribution to the excess noise. Finally, we also note that as far as the laser linewidth is concerned, the zero-frequency analysis presented in the previous section will remain valid, as long as the laser linewidth is much smaller than the excess noise bandwidth; this condition is generally fulfilled in practice [21].

This frequency-dependent behaviour of the excess noise was recently studied theoretically and demonstrated experimentally for the first time by A.M. van der Lee *et al.* [21], in the case of two non-orthogonal polarisation modes in a HeXe laser. The calculation presented here clearly shows that a decrease when increasing the analysis frequency is a general feature of the excess noise. Moreover, according to equation (26), a similar drop in the apparent value of the excess noise should be seen by looking at the spontaneous emission noise below threshold, as a function of the gain value: the highest the K value, the sharpest the drop. However, as said above, measuring and calibrating the output noise below threshold may be more difficult to realize than the convincing spectral analysis which was carried out in [21].

4 Conclusion

As a conclusion, we have presented a simple quantum calculation of the excess noise factor which apply to lasers with non-orthogonal eigenmodes. Within the approximations quoted above (homogeneous and uniform gain, mean field approximation), this result is demonstrated exactly and rather simply in a fully quantum framework. As was explained in reference [11], this quantum calculation does not introduce any "set of non-orthogonal eigenmodes", which would not permit a proper quantization. All calculations are based on the orthogonal sets $\{\hat{a}_{\rm in}\}$, $\{\hat{a}_{\rm out}\}$. On the other hand, the non-orthogonal combinations $PU^\dagger P\{\hat{a}_{\rm out}\}$, which represent the possible lasing modes, are generally non-commuting. This means that they are "incompatible" in a quantum-mechanical sense [4]: by hypothesis, only one of these modes can be lasing at a time (which one depends on the cavity parameters, see ref. [11]). Therefore, in a quantum framework, the non-orthogonal modes never appear as a "set": only one of them is used at a time, in order to determine a quantization basis in which the real modes actually remain orthogonal.

A last point which was raised recently [22] is whether very large values of K could provide a route towards a thresholdless laser. No such effect can appear in our linearized analysis, which applies to "macroscopic" lasers, where the threshold condition is always determined by the compensation of gain and loss. On the other hand, it appears from equation (27) that the number of spontaneous emission photons in the mode *at threshold* is actually increased by a factor which is just K. In our model, this effect is simply attributed to the presence of other modes, which are themselves close to threshold, and which leak into the mode of interest. The relevant physics for understanding the usual Petermann excess noise factor is thus the one of coupled laser oscillators, which does not say much about thresholdless lasing. We note however that both types of effects may mix up when considering a coupled-mode microcavity laser; a full quantum description of such a system remains an open question.

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Appendix: Loss-induced coupling

We give here some mathematical details about the definition of loss-induced coupling. The truncated matrix T (see eq. (14)) can be diagonalized in an orthogonal basis if and only if it is *normal*, *i.e.* $T^\dagger T = TT^\dagger$. This may occur even if T is non-unitary, *i.e.* in the presence of losses, and it is not so clear whether the excess noise effects should be attributed to losses, or to direct coupling between the laser modes, or to *loss-induced* coupling where laser modes are coupled via loss modes. We give below some arguments to support the loss-induced coupling approach.

In all cases, it is possible, as written above, to diagonalize T in a non-orthogonal basis $\{u_n\}$. Then one may pick up a vector in this basis, and iteratively build an orthogonal basis $\{w_n\}$ by constructing mutually orthogonal linear combinations of the $\{u_n\}$ (Schmidt orthogonalization procedure). For carrying out this procedure, the eigenvectors of T should be ordered by decreasing modulus of their eigenvalue, the last one being thus the mode with the lowest losses (lasing mode). Since T is diagonal in the $\{u_n\}$ basis, it is simple to show that it is triangular in the $\{w_n\}$ basis. The lasing mode may thus be coupled to all other (sub-threshold) modes.

Using this procedure, the basis for the laser modes is redefined through a unitary transform, in such a way that all direct coupling terms between the laser modes have been removed. Then the remaining non-diagonal terms in the triangular matrix represent precisely loss-induced coupling. On the other hand, losses which affect *independently* the (properly redefined) modes do not contribute to excess noise [11]. When defined in such a way, *i.e.* up to an arbitrary unitary transform within the laser modes, it appears than the existence of loss-induced coupling is a necessary and sufficient condition to get a non-normal truncated scattering matrix T: loss-induced coupling just means that there is no possible orthogonal basis where the triangular coupling matrix can be made diagonal.

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17. Outside the mean field approximation, this reference value will give the correct quantum value for the linewidth of a single-ended laser cavity (see [11], Appendix II). This value differs from the Schawlow-Townes value by the so-called longitudinal Petermann factor, which is unity in the mean field approximation.
18. This hypothesis corresponds to a single frequency operation of the laser, which is not always the case in present experiments (see *e.g.* ref. [10], which involves two polarization modes with different frequencies). The calculation presented in the present paper is not immediately applicable to such a situation, but the technique that we used can be extended straightforwardly to describe it.
19. For the sake of clarity, let us remind that outside the set of laser modes the matrices T, G are zero, while U, V are unity.
20. This is a hypothesis which amounts to say that there is no *gain-induced coupling* in the system, but only *loss-induced coupling* [11].
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